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CONJUGATION OF THE CHANNEL AND FILTRATION FLOWS OF A VISCOUS INCOMPRESSIBLE FLUID

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This paper is devoted to the problem of conjugation of high-velocity flows of a viscous fluid in wells or open channels and filtration flows in a porous medium. Usually in this case the fluid motion in a well (channel) is described by balance relationships [1-3] or by a hydraulic approximation using St. Venant's equations and their modifications [4]. This approach is based on the assumption that the relative velocity of conjugate flows is small. When the velocity is high enough, interaction between conjugate flows is possible only through an intermediate boundary layer close to the interface.

Below we present different variants of conjugation of such flows in the context of the boundary layer approximations for both flows.

In the latter case a class of self-similar regimes of flow is found and solvability of the boundary problems is established for mutually perpendicular boundary layers in a well and in an adjoining porous medium.

1. Statement of the Problem. The plain steady motion of an incompressible fluid in a well (channel) is governed by the Navier–Stokes equations

where u = (u, v) is the vector of the velocity of fluid flow of density $\rho = 1$; μ = const is the viscosity; $p = p_0 + \rho g h$, p_0 is the pressure; $g = g \nabla h$ is the vector of acceleration of gravity; F = 0.

The filtration fluid flow in a region D_2 adjoining to a region D_1 is described by the Navier–Stokes equations also, where, due to the assumption of the filtration theory, the resistance forces F are presented in the form $F = -\lambda u$, $\lambda(x, y) = m\mu k^{-1}$, where m is the porosity, and k is the permeability of the porous medium (ν = mu is the filtration velocity) ([1, p. 44-46] and [3, p. 159]). Note that the equations of the Navier–Stokes type are used in [2] to describe the fluid filtration in a granular media.

We restrict ourselves to the problems of conjugation of the filtration fluid flows in a porous medium (layer) and in a gallery of imperfect wells (a "plant well" or simply a well) [1-3, 5] corresponding to the vertical cross-section of a layer ($g = (-g, 0)$).

Let in the domains D_1 and D_2 the fluid flows be mostly directed along the OX and OY axes. Then in the conditions of approximation of the boundary layer we can use the following equations instead of the Navier–Stokes equations:

$$\mathbf{u} \nabla u = \mu u_{yy} - p_x,\ \nabla \cdot \mathbf{u} = 0,\ (x, y) \in D_1; \tag{1.1}$$

$$\mathbf{u} \nabla v = \mu v_{xx} - p_y - \lambda v,\ \nabla \cdot \mathbf{u} = 0,\ (x, y) \in D_2. \tag{1.2}$$

The vector u of the velocity of flow and the pressure p are considered to be continuous on the line of conjugation of the flows ($\Gamma = \bar{D}_1 \cap D_2$):

$$[\mathbf{u}] = 0,\ [p] = 0,\ (x, y) \in \Gamma. \tag{1.3}$$

Here $[f] = f|_{\Gamma_2} - f|_{\Gamma_1}$; $\Gamma_k = \Gamma \subset \partial D_k$ $(k = 1, 2)$; $f|_{\partial D_k}$ are the boundary values of f(x, y), $(x, y) \in \partial D_k$.

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Note that, taking into account the direction of the filtration flow after the substitution $x = \eta$, $y = -\xi$, $u = V$, $v = -U$, Eqs. (1.2) with $\lambda = 0$ transform into Prandtl's equations (1.1) of the boundary layer for $U(\xi, \eta)$, $V(\xi, \eta)$.

The experiments in [6] on fluid flows near the porous surfaces show the slipping effect, and a simple model is given to describe it.

Let for definiteness Γ: $y = 0$ be a sewing line and the domains D_1; $y < 0$, D_2: $y > 0$. Then similarly to [6] we can use instead of (1.3) the sewing conditions

$$[v] = [p] = 0; \frac{\partial u}{\partial y}\Big|_- = \frac{\alpha}{\sqrt{k}}(u - Q)\Big|_+, (x, y) \in \Gamma,$$

where $f|_\pm = f(x, \pm 0)$; $Q|_+$ is the liquid consumption through the porous surface; α is a constant characterizing the porous medium near Γ.

2. Conjugation of Filtration Flow and Free Flow on the Well's Wall. Let $D_1 = \{x > 0, 0 < y < h\}$ be the domain corresponding to the symmetric (about $y = h$) part of the well, and let $D_2 = \{x > 0, -H < y < 0\}$ be the domain of fluid filtration.

The boundary conditions for Eqs. (1.1), (1.2) in the domains D_1, D_2, are of the form

$$(u - u_0)|_{y=0} = 0,\ u_y|_{y=h} = 0,\ x \geqslant 0;\ u_x|_{x=0} = u_1(y) \geqslant 0,\ y \geqslant 0; \tag{2.1}$$

$$u|_{x=0} = 0,\ -H \leqslant y \leqslant 0;\ v|_{y=-H} = v_1(x),\ x \geqslant 0 \quad (u_0 = u(x, -0)). \tag{2.2}$$

For $p_y = C = \text{const}$, $y < 0$ a particular solution $u = 0$, $v = v_1(x)$ of the problem (1.2), (2.2) exists in the domain D_2, where $v_n(x)$ in (2.2) is defined as the solution of the problem

$$\mu v_1'' - \lambda v_1 - C = 0,\ v_1(0) = 0,\ v_1(\infty) = -C\lambda^{-1}.$$

Then for the problem (1.1), (2.1) in the well the first of the conditions (2.1) is of the form $u|_{y=0} = (0, v_1(x))$.

The following analogue of the problem of continuation of the boundary layer can be considered instead of (2.1) in the domain D_1:

$$(u - u_0)|_{y=0} = 0,\ u|_{y=h} = u_2(x),\ x \geqslant 0;\ u|_{x=0} = u_1(y) \geqslant 0,\ y \geqslant 0.$$

Here the velocity through the center of the well $u_2(x) > 0$ and the velocity profile at entry into the well $u_1(y) \geq 0$, $y \geq 0$ are considered as arbitrarily prescribed functions.

3. Conjugation of Filtration Flow and Free Flow at Entry into the Well. Let a layer be unsealed by the symmetric (about $y = 0$) well without deepening [5, p. 419], and, respectively,

$$D_1 = \{0 < y < h,\ 0 < x < X\},\ D_2 = \{x < 0,\ -H < y < h\}.$$

The flow through the domains D_1 and D_2 ($b = H$), can be described by the solution of the boundary problems

$$u|_{y=h} = (u_y, v)|_{y=0} = 0,\ x \geqslant 0;\ u|_{x=0} = u_0(y),\ y \geqslant 0; \tag{3.1}$$

$$(u - u_0)|_{x=0} = 0,\ y \geqslant 0;\ v|_{y=0} = v_1(x),\ x \leqslant 0 \tag{3.2}$$

for Eqs. (1.1), (1.2), respectively, where $u_0 = (u_0(y), v(+0, y))$ and $u_0(y)$, $v_1(x)$ are arbitrarily prescribed functions.

If at entry into the well the filtration flow is directed strictly along the well, i.e., $v|_{y=0} = 0$, then $u_0(y) = -(2\mu)^{-1}p_x(0)(h^2 - y^2)$ is uniquely determined from the solution of the boundary problem

$$\mu u_{0_{yy}} - p_x(0) = 0,\ u_0(h) = 0,\ u_0'(0) = 0.$$

Consider now the problem of sewing two boundary layers adjacent to the wall, when the straight line $\{y = 0, 0 \leq x \leq X\}$, is the well's wall, straight line $\{x = 0, y \leq 0\}$, is the impermeable roof of the layer, and $\{x = 0, 0 \leq y \leq h\}$ is an entry into the well.

According to this, there arise the following problems of continuation of the boundary layers (1.1) and (1.2) in the domains D_1 and D_2:

$$u|_{y=0} = 0,\ u|_{y=h} = u_1(x),\ 0 \leqslant x \leqslant X;\ u_x|_{x=0} = 0,\ y \geqslant 0; \tag{3.3}$$

$$(u - u_0(y), v)|_{x=0} = 0,\ v|_{y=-H} = v_1(x),\ x \leqslant 0; \tag{3.4}$$

$$\lim_{x\to-\infty} v(x, y) = v_\infty(y),\ -H \leqslant y \leqslant h; \tag{3.5}$$

$$v_\infty v_\infty' + p_y + \lambda v_\infty = 0,\ v_\infty(-H) = v_1(-\infty),\ y \geqslant -H. \tag{3.6}$$

Here $u_0(y) = 0$, $y \leq 0$, and when $y > 0$ $u_0 = (2\mu)^{-1}p_x(0)(y + u_1(0)yh^{-1}$ is determined from the solution of the boundary problem

$$\mu u_0'' - p_x(0) = 0,\ u_0(0) = 0,\ u_0(h) = u_1(0),$$

which is a consequence of (1.1), (3.3).

4. Self-Similar Solutions. We insert the current function $\psi(x, y)$ assuming that $u = \psi_y$, $v = -\psi_x$. Then Eqs. (1.1), (1.2) admit a self-similar solution of the form

$$\psi = y^k\varphi(\xi),\ \xi = y(nx+x_0)^{-1/n},\ x_0 = \text{const},$$

where the constants k and n are connected by the relation $k = n - 1$ for Eq. (1.1) and $k = 1 - n$ for Eq. (1.2).

Naturally, the conditions appear on the prescribed functions p(x, y) and λ(x, y) involve the equation:

$$p_x = \delta^{(1)}(nx + x_0)^{(k-3)/n} \text{ for } (1.1),\ \delta^{(1)} = \text{const};$$
$$p_y = \delta^{(2)}y^{k-3n},\ \lambda = -\delta_0^{(2)}y^{-2n} \text{ for } (1.2),\ (\delta^{(2)}, \delta_0^{(2)}) = \text{const}.$$

Then Eqs. (1.1), (1.2) transform into the quasilinear differential equations for the function $\varphi(\xi)$:

$$L^{(m)}\varphi \equiv \sum_{i=0}^{3} \lambda_i^{(m)} \frac{d^i\varphi}{d\xi^i} - f^{(m)} = 0\ (m = 1, 2). \tag{4.1}$$

Here

$$\lambda_3^{(m)} = \mu\ (m = 1, 2);\ \lambda_2^{(1)} = k(3\mu\xi^{-1} + \xi^k\varphi);$$
$$\lambda_1^{(1)} = 3\mu k(k - 1)\xi^{-2} + 2k\varphi\xi^{k-1} - (k - 1)\xi^k\varphi';\ \lambda_0^{(1)}$$
$$= \mu k(k - 1)\,(k - 2)\xi^{-3};$$
$$f^{(1)} = \delta^{(1)}\xi^{-k};\ \lambda_2^{(2)} = 3\mu(n + 1)\xi^{-1} + k\varphi\xi^{-(n+1)};\ \lambda_1^{(2)} = \mu(n+1)\,(2n$$
$$+ 1)\xi^{-2} + k(n + 1)\varphi\xi^{-(n+2)} - (k - n)\xi^{-(n+1)}\varphi' + \delta_0^{(2)}\xi^{-2(n+1)};$$
$$f^{(2)} = \delta^{(2)}\xi^{-3(n+1)};\ \lambda_0^{(2)} = 0.$$

The obvious consequences of the sewing conditions (1.3) should be fulfilled on the sewing line, and the characteristics of the outer flows should be specified when $\xi \to \pm\infty$, for example

$$(k\varphi + \xi\varphi')|_{\xi=\infty} = u_\infty^{(1)} \qquad \text{for } (1.1)\ ,$$
$$\xi^{n+1}\varphi'|_{\xi=-\infty} = v_\infty^{(2)} \qquad \text{for } (1.2)\ .$$

Consider the special case of sewing the self-similar solutions of Eqs. (1.1), (1.2), corresponding to a flow of the Poiseuille type in the open stream (the domain D_1 ($x > 0$)):

$$u = Cy^2,\ v = 0,\ p_x = 2C\mu,$$

so that in (4.1) $m = 1$, $k = 3$, $n = 4$, $\varphi(\xi) = C = \text{const}$.

Then Eq. (4.1) with $m = 2$, $k = 3$, $n = -2$, $\xi = y(-2x)^{1/2}$ ($x < 0$) corresponds to the filtration flow governed by Eq. (1.2).

The sewing conditions and the specification of the outer filtration flow lead to the boundary problem for Eq. (4.1):

$$\varphi(0) = \frac{1}{3}C,\ \varphi'(0) = 0,\ \xi^{-1}\varphi\big|_{\xi\to\infty} = C_0.$$

5. The Existence Theorem. Consider the problem (1.1), (1.2), (3.3)-(3.6). Functions $u_1(x)$, $v_1(x)$ in (3.3), (3.4) specified on ∂D ($D = D_1 \cup D_2$), $p = p_1(x)$, $x \geq 0$; $p = p_2(y)$, $y \leq 0$ ($p_1 = p_1(0)$, $y > 0$) in (1.1), (1.2), and the velocity $v_\infty(y)$ of the outer flow in a porous medium in (3.5), (3.6) are subject to the usual assumptions of the theory of a boundary layer [7-9]:

$$\begin{aligned}
&(p_1, p_2, u_1, v_1) \in C^{2+\alpha}(\partial D),\ \alpha > 0;\ p_1' < 0,\ x \geqslant 0;\\
&p_2' < 0,\ y \leqslant 0;\ v_\infty > 0,\ y \geqslant -H;\\
&(u_1, u_1') > 0,\ x > 0,\ u_1(0) > 0,\ u_1'(0) = 0;\\
&(v_1, v_1') > 0,\ x < 0,\ v_1(0) = 0,\ v_1'(0) > 0;\\
&\text{when } y = -H,\ x \to 0\ \mu v_1'' - p_2'(-H) - \lambda v_1 = O(x^2).
\end{aligned} \tag{5.1}$$

Here $f(x, y) \in C^{2+\alpha}(\partial D)$ if f and the second derivatives are bounded and Hölder continuous.

Theorem 5.1. Let the assumptions (5.1) be fulfilled. Then in the domain $D = D_1 \cup D_2$ $\forall (X, H) > 0$ for some $h > 0$ there exists a solution $u(x, y)$, $v(x, y)$ of the problem (1.1), (1.2), (3.3)-(3.6), which possesses the properties:

$$\begin{aligned}
&(u, u_y, u_{yy}) \in C(D_1),\ (v, u_x, v_y) \in C(\Omega_1) \qquad \forall\, \overline{\Omega}_1 \subset D_1;\\
&(v, v_x, v_{xx}) \in C(D_2),\ (u, u_x, v_y) \in C(\Omega_2) \qquad \forall\, \overline{\Omega}_2 \subset D_2;\\
&u(x, y) > 0,\ \frac{\partial u}{\partial y}\Big|_{y=0} \geqslant m_1 > 0,\ (x, y) \in D_1;\\
&v(x, y) > 0,\ \frac{\partial v}{\partial x}\Big|_{x=0} \geqslant m_2 > 0,\ (x, y) \in D_2.
\end{aligned}$$

The problem (1.1), (1.2), (3.3)-(3.6) falls into three problems, which are solved consequently in the domains D_1, $D_3 = \{-\infty < x < 0,\ -H < y \leq 0\}$ and $(D_2 \backslash \dot{D}_3)$.

In the domain D_1 functions $u(x, y)$, $v(x, y)$ satisfy the problem (1.1), (3.3) for which the existence theorem stated above is proved like Theorem 1 in [7].

In the domain D_3 there arises the problem of continuation of the boundary layer (1.2) solvable for any $\forall H > 0$ [7, 8] by virtue of the assumption $p_2'(y) < 0$, $y < 0$.

Note that the velocity profile $v_2 = v(x, y)|_{y=0}$, $x < 0$ resulting from the solution of the problem in D_3 has all the properties of $v_1(x)$ in (5.1).

To find the solution $u(x, y)$, $v(x, y)$, $(x, y) \in (D_2 \backslash D_3)$ of the problem (1.2), (3.4)-(3.6) we transfer to Misses' variables:

$$y = y,\ \psi = \psi(x, y),\ v = -\psi_x,\ u - u_0(y) = \psi_y,$$

In this case the condition $v|_{y=-H} = v_1(x)$ in (3.4) is replaced by $v|_{y=0} = v_2(x)$. As a result we get for $\omega = v^2$

$$\omega_y - u_0(y)\omega_\psi = \mu\sqrt{\omega}\omega_{\psi\psi} - 2\lambda\sqrt{\omega},\ (\psi, y) \in \Omega, \tag{5.2}$$

where $\Omega = \{-\infty < \psi < 0, 0 < y < h\}$ is a mapping of $(D_2 \backslash D_3)$. The conditions (3.4)-(3.6) transform into the following:

$$\omega|_{\psi=0} = 0,\ \omega|_{y=0} = \omega_0(\psi),\ \lim_{\psi\to-\infty} \omega = v_\infty^2,$$
$$v_\infty' = -\lambda,\ v_\infty(0) = v_2(-\infty),\ \omega_0(\int_x^0 v_2(\tau)d\tau) = v_2^2(x).$$

Upon differentiating, the term $2\lambda\sqrt{\omega}$ of Eq. (5.2) gives a function unbounded in $\lambda > 0$, no essential changes are needed to prove the existence of the solution of the problem (1.2), (3.4)-(3.6) in comparison with the case when $\lambda = 0$ [7, 8].

Remark 5.1. The flow in a well may also be simulated by a Poiseuille flow.

Remark 5.2. It is not difficult to state problems similar to those from points 2, 3 for different models of nonhomogeneous fluid [10, 11] also.

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REFERENCES

1. P. Ya. Polubarinova-Kochina, Theory of Motion of Ground Waters [in Russian], Nauka, Moscow (1977).
2. I. V. Shirko, "Numerical research of flows in granular media," in: Numerical Simulation in Aerohydrodynamics [in Russian], Nauka, Moscow (1986).
3. The Development of Investigations of Filtration Theory in the USSR (1917-1967) [in Russian], Nauka, Moscow (1969).
4. A. F. Voevodin and S. M. Shugrin, Numerical Methods of Computation of One-dimensional Systems [in Russian], Nauka, Novosibirsk (1981).
5. J. Bear, D. Zaslavsky, and S. Irmay, Foundations of Water Filtration [Russian translation], Mir, Moscow (1971).
6. C. S. Beavers and D. D. Joseph, "Boundary conditions at a naturally permeable wall," J. Fluid Mech., **30**, No. 1, 197 (1967).
7. O. A. Oleinik, "On a system of equations of the theory of a boundary layer," Vychisl. Metody Mat. Fiz., **3**, No. 3 (1963).
8. A. N. Suslov, "On a system of equations of the magnetohydrodynamic boundary layer," Vestn. MGU, Mat., Mech., **1**, No. 2 (1974).
9. N. V. Khusnutdinova, "Heat boundary layer on a plate," Dokl. Akad. Nauk, **285**, No. 3 (1985).
10. S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, Boundary Problems in Mechanics of Nonhomogeneous Fluids, Nauka, Novosibirsk (1983).
11. V. N. Monakhov, "Mathematical pattern of filtration of a nonhomogeneous fluid," in: Dynamics of Continuous Media, Collection of Scientific Works, Lavrentyev Institute, No. 90 (1989).